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q-regular variation and q-difference equations

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Abstract

We introduce the concept of q-regularly varying functions and establish basic properties of such functions. Among other things it is shown that considering regular variation in q-calculus is somehow natural and leads to interesting observations and simplifications compared with classical continuous and discrete theories. The obtained theory is applied to an investigation of asymptotic behavior of solutions to linear second-order q-difference equations.

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1. Introduction

We are interested in obtaining a basic theory of the so-called q-regularly varying functions, i.e., the functions defined on the lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ (or on $q^{\mathbb{Z}}$), $q > 1$, with regularly varying like behavior. As we shall see, this setting is very natural for examining regular variation and leads to interesting observations and simplifications compared with classical continuous and discrete theories. In the second part of the paper, we apply this theory to study asymptotic behavior of solutions to the second-order linear q-difference equation $\mathrm{D}_q^2 y(t) = p(t)y(qt)$.

Recall that a measurable *function* $f : [a, \infty) \to (0, \infty)$ is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\vartheta} \qquad \text{for all} \quad \lambda > 0; \tag{1}$$

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we write $f \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *slowly varying*. Fundamental properties of regularly varying functions are that relation (1) holds uniformly on each compact λ set in $(0, \infty)$, and $f \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$ if and only if it may be written in the form $f(x) = \varphi(x)x^{\vartheta} \exp\left\{\int_a^x \eta(s)/s \, ds\right\}$, where φ and η are measurable with $\varphi(x) \to C \in (0, \infty)$ and $\eta(x) \to 0$ as $x \to \infty$, see [2, 8, 11, 17]. In the basic theory of regularly varying sequences two main approaches are known. First, the approach by Karamata [10], based on a counterpart of the continuous definition: a positive *sequence* $\{f_k\}$, $k \in \{a, a+1, \ldots\} \subset \mathbb{Z}$, is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if

$$\lim_{k\to\infty} \frac{f_{[\lambda k]}}{f_k} = \lambda^{\vartheta} \qquad \text{for all} \quad \lambda > 0, \tag{2}$$

where $[u]$ denotes the integer part of u. Second, the approach by Galambos and Seneta [7], based on a purely sequential definition: a positive *sequence* $\{f_k\}$ is said to be *regularly varying of index* ϑ if there exists a positive sequence $\{\alpha_k\}$ satisfying $f_k \sim C\alpha_k$ and $\lim_{k\to\infty} k(1 - \alpha_{k-1}/\alpha_k) = \vartheta$, with C being a positive constant. In [6] it was shown that these two definitions are equivalent. In [13] it is suggested to replace the second condition in the latter definition (equivalently) by $\lim_{k\to\infty} k\Delta\alpha_k/\alpha_k = \vartheta$. A regularly varying sequence can be represented as $f_k = \varphi_k k^{\vartheta} \prod_{j=a}^{k-1}(1 + \psi_j/j)$, see [13], or as $f_k = \varphi_k k^{\vartheta} \exp\left\{\sum_{j=a}^{k-1} \psi_j/j\right\}$, where $\varphi_k \to C \in (0, \infty)$ and $\psi_k \to 0$ as $k \to \infty$, see [6, 7]. The so-called imbedding theorem, see [6, 7], enables us to apply the continuous theory in the theory of regularly varying sequences. Recall that the theory of regular variation can be viewed as the study of relations similar to (1) or (2), together with their wide applications, see, e.g., [2, 8, 12–14]. There is a very practical way how regularly varying functions can be understood: extension in a logical and useful manner of the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies 'more slowly' than a power function. In [15, 16] we introduced the concept of regular variation on so-called time scales (calculus is made on a closed subset of reals), which offers something more than the imbedding result: once a result on a general time scale is proved, it automatically holds for the continuous and the discrete case, without any other effort. Moreover, at the same time, the theory also works on other time scales which may be different from the 'classical' ones. However, that theory requires certain additional condition on the graininess (i.e., the 'distance' between two neighboring points) of a time scale, namely the graininess has to be sufficiently small. It is demonstrated in [16] that if such a condition is violated (e.g., graininess is like in $q^{\mathbb{N}_0}$), then the theory fails to hold. In this paper, we show how the approach should be modified for the theory of regular variation to work reasonably also in q-calculus and q-regularly varying functions somehow kept the above described properties. Moreover, as we shall see, considering the case $q^{\mathbb{N}_0}$ in the theory of regular variation is actually very natural and leads to surprising observations and simplifications. Applications of q-regular variation in the qualitative theory of second-order linear q-difference equations are also given. For related results concerning asymptotics of differential and difference equations see [12, 14], respectively.

This paper is organized as follows. In the following section we first recall basic concepts of q-calculus. Then we introduce the concept of q-regular variation. We also establish various characterizations of q-regular variation and show relations (an imbedding result) with the continuous theory of regular variation. Some important basic properties of q-regularly varying functions are established as well. In section 3 we apply the obtained theory: we establish necessary and sufficient conditions for all positive solutions of the second-order linear q-difference equation $D_q^2 y(t) = p(t)y(qt)$ to be q-regularly varying with a known index.

2. *q*-regular variation

We start with some preliminaries on q-calculus. For some details on this topic see [1, 9]. See also [4] for the calculus on time scales which contains q-calculus. The q-derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $\mathrm{D}_q f(t) = [f(qt) - f(t)]/[(q-1)t]$. Here are some useful rules: $\mathrm{D}_q(fg)(t) = g(qt)\mathrm{D}_q f(t) + f(t)\mathrm{D}_q g(t) = f(qt)\mathrm{D}_q g(t) + g(t)\mathrm{D}_q f(t)$, $\mathrm{D}_q(f/g)(t) = [g(t)\mathrm{D}_q f(t) - f(t)\mathrm{D}_q g(t)]/[g(t)g(qt)]$, $f(qt) = f(t) + (q-1)t\mathrm{D}_q f(t)$. The q-integral of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$\int_a^b f(t)\,\mathrm{d}_q t = \begin{cases} (q-1)\sum_{t\in[a,b)\cap q^{\mathbb{N}_0}} tf(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ (1-q)\sum_{t\in[b,a)\cap q^{\mathbb{N}_0}} tf(t) & \text{if} \quad a > b, \end{cases}$$

$a, b \in q^{\mathbb{N}_0}$. The improper q-integral is defined by $\int_a^\infty f(t)\,\mathrm{d}_q t = \lim_{b\to\infty}\int_a^b f(t)\,\mathrm{d}_q t$. Since the fraction $(q^a - 1)/(q-1)$ appears quite frequently, let us introduce the notation $[a]_q = (q^a - 1)/(q-1)$ for $a \in \mathbb{R}$. Note that $\lim_{q\to 1}[a]_q = a$. For $p \in \mathcal{R}$ (i.e., for $p : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfying $1 + (q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$) we denote $e_p(t,s) = \prod_{u\in[s,t)\cap q^{\mathbb{N}_0}}[(q-1)up(u) + 1]$ for $s < t$, $e_p(t,s) = 1/e_p(s,t)$ for $s > t$, and $e_p(t,t) = 1$, where $s, t \in q^{\mathbb{N}_0}$. Here are some useful properties of $e_p(t,s)$: for $p \in \mathcal{R}$, $e(\cdot, a)$ is a solution of the IVP $\mathrm{D}_q y = p(t)y$, $y(a) = 1$, $t \in q^{\mathbb{N}_0}$. If $s \in q^{\mathbb{N}_0}$ and $p \in \mathcal{R}^+$, where $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + (q-1)tp(t) > 0 \text{ for all } t \in q^{\mathbb{N}_0}\}$, then $e_p(t,s) > 0$ for all $t \in q^{\mathbb{N}_0}$. If $p, r \in \mathcal{R}$, then $e_p(t,s)e_p(s,u) = e_p(t,u)$ and $e_p(t,s)e_r(t,s) = e_{p+r+t(q-1)pr}(t,s)$.

Now we are ready to introduce the concept of q-regular variation.

Definition 1. *A function* $f : q^{\mathbb{N}_0} \to (0,\infty)$ *is said to be q-regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, *if there exists a function* $\alpha : q^{\mathbb{N}_0} \to (0,\infty)$ *satisfying*

$$f(t) \sim C\alpha(t), \qquad \text{and} \qquad \lim_{t\to\infty}\frac{t\mathrm{D}_q\alpha(t)}{\alpha(t)} = [\vartheta]_q, \tag{3}$$

with C being a positive constant. If $\vartheta = 0$, *then* f *is said to be q-slowly varying.*

The totality of q-regularly varying functions of index ϑ is denoted by $\mathcal{RV}_q(\vartheta)$. The totality of q-slowly varying functions is denoted by $\mathcal{SV}_q$. In fact, we have defined q-regular variation at infinity. If we consider a function $f : q^{\mathbb{Z}} \to (0,\infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(\cdot)$ is said to be q-regularly varying at zero if $f(1/t)$ is q-regularly varying at infinity. But it is apparent that it is sufficient to develop just the theory of q-regular variation at infinity. It is easy to see that the function t^ϑ is a typical representative of the class $\mathcal{RV}_q(\vartheta)$. Of course, this class is much wider as can be seen from the representations derived in the following theorem, where we also offer some other (simple) characterizations of q-regular variation and show a relation with the continuous theory.

Theorem 1.

(i) (Simple characterization) For a positive function f, $f \in \mathcal{RV}_q(\vartheta)$ *if and only if* f *satisfies*

$$\lim_{t\to\infty}\frac{f(qt)}{f(t)} = q^\vartheta. \tag{4}$$

Moreover, $f \in \mathcal{RV}_q(\vartheta)$ *if and only if* f *satisfies just the latter condition in (3), i.e.,* $\lim_{t\to\infty} t\mathrm{D}_q f(t)/f(t) = [\vartheta]_q$.

(ii) (Zygmund-type characterization) For a positive function f, $f \in \mathcal{RV}_q(\vartheta)$ *if and only if* $f(t)/t^\gamma$ *is eventually increasing for each* $\gamma < \vartheta$ *and* $f(t)/t^\eta$ *is eventually decreasing for each* $\eta > \vartheta$.

(iii) (Representation I) $f \in \mathcal{RV}_q(\vartheta)$ *if and only if* f *has the representation*

$$f(t) = \varphi(t)e_\delta(t, 1), \tag{5}$$

where $\varphi : q^{\mathbb{N}_0} \to (0, \infty)$ *tends to a positive constant and* $\delta : q^{\mathbb{N}_0} \to \mathbb{R}$ *satisfies* $\lim_{t\to\infty} t\delta(t) = [\vartheta]_q$ *and* $\delta \in \mathcal{R}^+$*. Without loss of generality, in particular in the only if part, the function* φ *in (5) can be replaced by a positive constant.*

(iv) (Representation II) $f \in \mathcal{RV}_q(\vartheta)$ *if and only if* f *has the representation*

$$f(t) = t^\vartheta \tilde{\varphi}(t)e_\psi(t, 1), \tag{6}$$

where $\tilde{\varphi} : q^{\mathbb{N}_0} \to (0, \infty)$ *tends to a positive constant and* $\psi : q^{\mathbb{N}_0} \to \mathbb{R}$ *satisfies* $\lim_{t\to\infty} t\psi(t) = 0$ *and* $\psi \in \mathcal{R}^+$*. Without loss of generality, in particular in the only if part, the function* $\tilde{\varphi}$ *in (6) can be replaced by a positive constant.*

(v) (Karamata-type characterization) For a positive function f, $f \in \mathcal{RV}_q(\vartheta)$ *if and only if* f *satisfies*

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = (\tau(\lambda))^\vartheta \qquad \text{for} \quad \lambda \geqslant 1, \tag{7}$$

where $\tau : [1, \infty) \to q^{\mathbb{N}_0}$ *is defined as* $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \leqslant x\}$.

(vi) (Imbeddability) If $f \in \mathcal{RV}_q(\vartheta)$ *then* $R \in \mathcal{RV}(\vartheta)$*, where*

$$R(x) = f(\tau(x)) \left(\frac{x}{\tau(x)}\right)^\vartheta \qquad \text{for} \quad x \in [1, \infty). \tag{8}$$

Conversely, if $R \in \mathcal{RV}(\vartheta)$*, then* $f \in \mathcal{RV}_q(\vartheta)$*, where* $f(t) = R(t)$ *for* $t \in q^{\mathbb{N}_0}$.

Proof. (i) If $f \in \mathcal{RV}_q(\vartheta)$, then with $\lim_{t\to\infty} \varphi(t) = C > 0$ we have

$$\lim_{t\to\infty} \frac{f(qt)}{f(t)} = \lim_{t\to\infty} \frac{\varphi(qt)\alpha(qt)}{\varphi(t)\alpha(t)} = \lim_{t\to\infty} \frac{\alpha(t) + (q-1)tD_q\alpha(t)}{\alpha(t)} = 1 + (q-1)[\vartheta]_q = q^\vartheta,$$

which implies (4). Conversely,

$$\lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = \lim_{t\to\infty} \frac{t}{t(q-1)} \left(\frac{f(qt)}{f(t)} - 1\right) = [\vartheta]_q.$$

(ii) If $f \in \mathcal{RV}_q(\vartheta)$, then by (i)

$$\frac{f(qt)}{(qt)^\gamma} - \frac{f(t)}{t^\gamma} = \frac{f(t)}{(qt)^\gamma} \left(\frac{f(qt)}{f(t)} - q^\gamma\right) = \frac{f(t)}{(qt)^\gamma}(q^\vartheta - q^\gamma + o(1)).$$

The monotonicity for large t with η instead of γ follows similarly. Conversely, the monotonicities imply $(qt/t)^\gamma \leqslant f(qt)/f(t) \leqslant (qt/t)^\eta$ so that $q^\gamma \leqslant f(qt)/f(t) \leqslant q^\eta$. The statement follows by choosing γ and η arbitrarily close to ϑ and using (i).

Statements (iii) and (iv) follow from the implications $f \in \mathcal{RV}_q(\vartheta) \Rightarrow f$ satisfies (5) $\Rightarrow f$ satisfies (6) $\Rightarrow f \in \mathcal{RV}_q(\vartheta)$, which will be proved next. If $f \in \mathcal{RV}_q(\vartheta)$, then there is δ such that $\mathrm{D}_q\alpha(t) = \delta(t)\alpha(t)$ and $\lim_{t\to\infty} t\delta(t) = [\vartheta]_q$. Since this is a first-order q-difference equation and α is its positive solution, it has the form $\alpha(t) = \alpha_0 e_\delta(t, 1)$ with $\alpha_0 > 0$. Formula (5) now follows from the first condition in (3) and the fact that $e_\delta(t, 1) > 0$ implies $\delta \in \mathcal{R}^+$. If f satisfies (5), then we have $f(t) = \varphi(t)t^\vartheta L(t)$, where $L(t) = e_\delta(t, 1)/t^\vartheta > 0$ and $\lim_{t\to\infty} t\delta(t) = [\vartheta]_q$. We show that $\lim_{t\to\infty} tD_q L(t)/L(t) = 0$. Indeed, from

$$D_q L(t) = \frac{\delta(t)e_\delta(t, 1) - e_\delta(t, 1)[\vartheta]_q}{q^\vartheta t^\vartheta}$$

we obtain

$$\frac{tD_qL(t)}{L(t)} = \frac{t\delta(t)}{q^\vartheta} - \frac{[\vartheta]_q}{q^\vartheta} \to 0$$

as $t \to \infty$. Hence, arguing as in the previous part, there is ψ such that $L(t) = \psi_0 e_\psi(t, 1) > 0$ where $\psi_0 > 0$ and $\lim_{t\to\infty} t\psi(t) = 0$. Thus, f can be written in the form (6). If f satisfies (6), we have $f(t) = \tilde{\varphi}(t)\alpha(t)$, where $\alpha(t) = t^\vartheta e_\psi(t, 1) > 0$ and $\lim_{t\to\infty} t\psi(t) = 0$. Similarly as in the previous part, it is easy to show that $\lim_{t\to\infty} tD_q\alpha(t)/\alpha(t) = [\vartheta]_q$. The fact that φ and $\tilde{\varphi}$ can be replaced by a constant follows from (i).

(v) The if part trivially follows from (i). Conversely assume that $f \in \mathcal{RV}_q(\vartheta)$. Then (6) holds. First observe that if $t \in q^{\mathbb{N}_0}$ and $\lambda \geqslant 1$, then $t = q^n$ and $\lambda \in [q^j, q^{j+1})$ for some $j, n \in \mathbb{N}_0$. Hence, $\tau(\lambda t)/t = q^{n+j}/q^j = \tau(\lambda)$. From (6) we have

$$\frac{f(\tau(\lambda t))}{f(t)} = \frac{\tilde{\varphi}(\tau(\lambda t))}{\tilde{\varphi}(t)} \left(\frac{\tau(\lambda t)}{t}\right)^\vartheta e_\psi(\tau(\lambda t), t).$$

Hence,

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = (\tau(\lambda))^\vartheta \lim_{t\to\infty} e_\psi(\tau(\lambda t), t).$$

Set $q^n = t, q^{n+m} = \tau(\lambda t), m, n \in \mathbb{N}_0$. Note that $\tau(\lambda) = q^m$, where m is fixed since $\tau(\lambda t) = \tau(\lambda)t$. We have $e_\psi(\tau(\lambda t), t) = \prod_{j=n}^{n+m-1}[(q-1)q^j\psi(q^j) + 1]$. Since $q^j\psi(q^j) \to 0$ as $j \to \infty$, we obtain $\lim_{t\to\infty} e_\psi(\tau(\lambda t), t)$. Hence (7) holds for $\lambda \geqslant 1$.

(vi) First we show that if f satisfies (7), then $R : [1, \infty) \to (0, \infty)$ given by (8) satisfies $R \in \mathcal{RV}(\vartheta)$. Note that $R(t) = f(t)$ for $t \in q^{\mathbb{N}_0}$. We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} &= \lim_{x\to\infty} \frac{f(\tau(\lambda x))}{f(\tau(x))} \left(\frac{\lambda x}{\tau(\lambda x)}\right)^\vartheta \left(\frac{\tau(x)}{x}\right)^\vartheta \\
&= \lambda^\vartheta \lim_{x\to\infty} \frac{f(\tau(\lambda\tau(x)))}{f(\tau(x))} \Omega(x, \lambda) \\
&= \lambda^\vartheta (\tau(\lambda))^\vartheta \lim_{x\to\infty} \Omega(x, \lambda),
\end{aligned}$$

where

$$\Omega(x, \lambda) = \left(\frac{\tau(x)}{\tau(\lambda x)}\right)^\vartheta \frac{f(\tau(\lambda x))}{f(\tau(\lambda\tau(x)))}.$$

Since for each $\lambda, x \geqslant 1$, there are $m, n \in \mathbb{N}_0$ such that $\lambda \in [q^m, q^{m+1})$ and $x \in [q^n, q^{n+1})$, we have $\lambda x \in [q^{m+n}, q^{m+n+2})$, and so either (I) $\tau(\lambda x) = q^{m+n} = \tau(\lambda)\tau(x)$ or (II) $\tau(\lambda x) = q^{m+n+1} = q\tau(\lambda)\tau(x)$. Recall $\tau(\lambda\tau(x)) = \tau(\lambda)\tau(x)$. In case (I) we obtain $\Omega(x, \lambda) = (\tau(\lambda))^{-\vartheta}$, while in case (II) $\Omega(x, \lambda) = (q\tau(\lambda))^{-\vartheta} f(q\tau(\lambda)\tau(x))/f(\tau(\lambda)\tau(x))$. Since $\lim_{t\to\infty} f(qt)/f(t) = q^\vartheta$, from (I) and (II) we obtain $\lim_{x\to\infty} \Omega(x, \lambda) = (\tau(\lambda))^{-\vartheta}$. Hence, $\lim_{x\to\infty} R(\lambda x)/R(x) = \lambda^\vartheta$ for all $\lambda > 1$ and so by [2, theorem 1.4.1], for all $\lambda > 0$. Consequently, $R \in \mathcal{RV}(\vartheta)$. Conversely, if $R \in \mathcal{RV}(\vartheta)$, then by [2, theorems 1.3.1, 1.4.1], $R(x) = \Phi(x)x^\vartheta \exp\left\{\int_1^x \Psi(s)/s\, \mathrm{d}s\right\}$, where Φ, Ψ are bounded measurable functions on $[1, \infty)$ such that $\lim_{x\to\infty} \Phi(x) = \text{const} > 0$ and $\lim_{x\to\infty} \Psi(x) = 0$ (Ψ may be taken as continuous). Hence for $t \in q^{\mathbb{N}_0}$ we have $f(t) = \Phi(t)t^\vartheta \exp\left\{\int_1^t \Psi(s)/s\, \mathrm{d}s\right\}$. Then,

$$\frac{f(qt)}{f(t)} = \frac{\Phi(qt)}{\Phi(t)} q^\vartheta \exp\left\{\int_t^{qt} \Psi(s)/s\, \mathrm{d}s\right\}.$$

Using the mean value theorem, $\int_t^{qt} \Psi(s)/s\, \mathrm{d}s = \Psi(\omega(t)) \ln q \to 0$ as $t \to \infty$, where $t \leqslant \omega(t) \leqslant qt$. Consequently $\lim_{t\to\infty} f(qt)/f(t) = q^\vartheta$, and the statement follows from (i).

□

Remark 1. (i) (Important) The so-called normalized regularly varying functions of index ϑ can be defined as those satisfying the second condition in (3) or, equivalently, as those having representation (5) or (6) with a constant instead of $\varphi(t)$ or $\tilde{\varphi}(t)$, respectively. However, in contrast to the classical continuous or discrete case, owing to (i) and (ii) of theorem 1, the distinction between normalized (or Zygmund) and ordinary regular variation disappears in q-calculus. Therefore, we do not need to introduce the concept of a normalized regular variation. Moreover, in the q-calculus case we have another property not known in the classical theories: a Karamata-type characterization (7) can be substantially simplified to (4). Note that for the discrete case, an analog of (7) is $f([\lambda t])/f(t) \to \lambda^{\varrho}$ and an analog for (4) can be seen as $f(t+1)/f(t) \to 1$. However, the latter one is just necessary for regular variation on $\mathbb{Z}$. Altogether we see that regularly varying functions in q-calculus can be defined very simply by (4) or by the second condition in (3), and that $\varphi(t)$ and $\tilde{\varphi}(t)$ in representations (5) and (6), respectively, can be replaced by a positive constant without loss of generality. The reason for this simplification may be that regular variation can be based on a product characterization which is very natural for the q-calculus case.

(ii) A suitable extension of the operator τ enables us to have formula (7) also for $\lambda \in (0, 1)$.

(iii) Observe how the above (but also subsequent) results nicely resemble continuous results as $q \to 1$.

Regularly varying functions on $q^{\mathbb{N}_0}$ possess a number of properties. We list the following ones which will be needed later.

Proposition 1. *Regularly varying functions have the following properties:*

(i) It holds $f \in \mathcal{RV}_q(\vartheta)$ *iff* $f(t) = t^{\vartheta}L(t)$*, where* $L \in \mathcal{SV}_q$.

(ii) Let $f \in \mathcal{RV}_q(\vartheta)$*. Then* $\lim_{t\to\infty} \log f(t)/\log t = \vartheta$*. This implies* $\lim_{t\to\infty} f(t) = 0$ *if* $\vartheta < 0$ *and* $\lim_{t\to\infty} f(t) = \infty$ *if* $\vartheta > 0$.

(iii) Let $f \in \mathcal{RV}_q(\vartheta)$*. Then* $\lim_{t\to\infty} f(t)/t^{\vartheta-\varepsilon} = \infty$ *and* $\lim_{t\to\infty} f(t)/t^{\vartheta+\varepsilon} = 0$ *for every* $\varepsilon > 0$.

(iv) Let $f \in \mathcal{RV}_q(\vartheta)$*. Then* $f^{\gamma} \in \mathcal{RV}_q(\gamma\vartheta)$.

(v) Let $f \in \mathcal{RV}_q(\vartheta_1)$ *and* $g \in \mathcal{RV}_q(\vartheta_2)$*. Then* $fg \in \mathcal{RV}_q(\vartheta_1 + \vartheta_2)$ *and* $1/f \in \mathcal{RV}_q(-\vartheta_1)$.

(vi) Let $f \in \mathcal{RV}_q(\vartheta)$*. Then* f *is decreasing provided* $\vartheta < 0$*, and it is increasing provided* $\vartheta > 0$*. A concave* f *is increasing. If* $f \in \mathcal{SV}_q$ *is convex, then it is decreasing.*

Proof. (i), (iv), (v) The proofs of these parts are trivial.

(ii) From (5), using the q-L'Hospital rule, we have

$$\lim_{t\to\infty} \frac{\log f(t)}{\log t} = \lim_{t\to\infty} \frac{\sum_{s\in[1,t)\cap q^{\mathbb{N}_0}} \log[(q-1)s\delta(s)+1]}{\log t} = \lim_{t\to\infty} \frac{\log[(q-1)t\delta(t)+1]}{\log q} = \vartheta.$$

Alternatively we can see it from the imbedding result.

(iii) Follows from (6) and (ii) of this proposition.

(vi) The part for $\vartheta \neq 0$ is simple. For $\vartheta = 0$, i.e., $f \in \mathcal{SV}_q$, first we show that $D_q^2 f(t) > 0$ implies eventual monotonicity of f. Indeed, either we have $D_q f(t) < 0$ for all $t \in q^{\mathbb{N}_0}$, or if there is $t_0 \in q^{\mathbb{N}_0}$ such that $D_q f(t_0) \geqslant 0$, then $0 \leqslant D_q f(t_0) < D_q f(qt_0) < \cdots$, hence $D_q f(t) > 0$ for all $t \in (t_0, \infty) \cap q^{\mathbb{N}_0}$. By a contradiction assume that $D_q f(t) \geqslant 0$. Thanks to the convexity we have $D_q f(t) \geqslant M > 0$ for large $t \in q^{\mathbb{N}_0}$ and for some $M > 0$. Integrating from s to t we obtain $f(t) \geqslant f(s) + (t-s)M$. But now f cannot be slowly varying by (iii) of this proposition. □

3. q-regular variation and second-order q-difference equations

Consider the second-order linear q-difference equation

$$\mathrm{D}_q^2 y(t) = p(t)y(qt), \tag{9}$$

$t \in q^{\mathbb{N}_0}$, where $p : q^{\mathbb{N}_0} \to (0, \infty)$. Qualitative and quantitative properties of equations of this type were studied, e.g., in [1, 3–5].

Recall that all nontrivial solutions of (9) are nonoscillatory (i.e., are eventually of one sign) and eventually monotone. Because of linearity, without loss of generality, it is sufficient to consider just positive solutions of (9). Next we establish necessary and sufficient conditions for all positive solutions of (9) to be q-regularly varying.

Theorem 2. *(i) Equation (9) has a fundamental set of solutions*

$$u(t) = L(t) \in \mathcal{SV}_q, \qquad v(t) = t\tilde{L}(t) \in \mathcal{RV}_q(1) \tag{10}$$

if and only if

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}_q s = 0. \tag{11}$$

Moreover, $\tilde{L} \in \mathcal{SV}_q$ *with* $\tilde{L}(t) \sim 1/L(t)$. *All positive decreasing solutions of (9) belong to* $\mathcal{SV}_q$ *and all positive increasing solutions of (9) belong to* $\mathcal{RV}_q(1)$. *Any of two conditions in (10) implies (11).*

(ii) Equation (9) has a fundamental set of solutions

$$u(t) = t^{\vartheta_1}L(t) \in \mathcal{RV}_q(\vartheta_1), \quad v(t) = t^{\vartheta_2}\tilde{L}(t) \in \mathcal{RV}_q(\vartheta_2) \tag{12}$$

if and only if

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}_q s = A > 0, \tag{13}$$

where $\vartheta_i = \log_q[(q-1)\lambda_i + 1], i = 1, 2, \lambda_1 < 0 < \lambda_2$, *are the roots of the equation* $\lambda^2 - [A(q-1)+1]\lambda - A = 0$. *It holds* $\vartheta_1 < 0 < \vartheta_2$, $\lambda_2 = [\vartheta_2]_q = A(q-1)+1-[\vartheta_1]_q = A(q-1)+1-\lambda_1$ *and* $\vartheta_2 = 1-\vartheta_1$. *Moreover,* $L, \tilde{L} \in \mathcal{SV}_q$ *with* $\tilde{L}(t) \sim 1/(q^{\vartheta_1}[1-2\vartheta_1]_q L(t))$. *All positive decreasing solutions of (9) belong to* $\mathcal{RV}_q(\vartheta_1)$ *and all positive increasing solutions of (9) belong to* $\mathcal{RV}_q(\vartheta_2)$. *Any of two conditions in (12) implies (13).*

Proof. Parts (i) and (ii) will be proved simultaneously assuming $A \geqslant 0$ in (13) and, consequently, $\lambda_1 \leqslant 0$ or $\vartheta_1 \leqslant 0$, if it is not said otherwise. We will use the notation $[a,\infty)_q = \{a, aq, aq^2, \ldots\} \subseteq q^{\mathbb{N}_0}$.

Necessity. Let $u \in \mathcal{RV}_q(\vartheta_1)$ be a positive decreasing solution of (9) on $[a,\infty)_q$. Set $w = D_q u/u$. Then $w(t) < 0$ and satisfies the Riccati-type q-difference equation

$$\mathrm{D}_q w(t) - p(t) + \frac{w^2(t)}{1+(q-1)tw(t)} = 0 \tag{14}$$

with $w \in \mathcal{R}^+$ on $[a,\infty)_q$. We have $\lim_{t\to\infty} tw(t) = [\vartheta_1]_q$ and so $\lim_{t\to\infty} w(t) = 0$. We show that $\int_a^\infty w^2(t)/(1+(q-1)tw(t))\,\mathrm{d}_q t < \infty$. Since $1+(q-1)tw(t) \to q^{\vartheta_1}$, we have $1+(q-1)tw(t) > q^{\vartheta_1}/2$ for large t. Moreover, there is $N > 0$ such that $|w(t)| \leqslant N/t$ for large t. Without loss of generality, these large t's can be taken as $t \in [a,\infty)_q$. Then

$$\int_a^\infty \frac{w^2(t)}{1+(q-1)tw(t)}\,\mathrm{d}_q t \leqslant \frac{2N^2 q}{q^{\vartheta_1}} \int_a^\infty \frac{\mathrm{d}_q t}{qt^2} = \frac{2N^2 q}{aq^{\vartheta_1}},$$

since $D_q(1/t) = -1/(qt^2)$. Integration of (14) and multiplication by t yield

$$-tw(t) + t\int_t^\infty \frac{w^2(s)}{1+(q-1)sw(s)}\,\mathrm{d}_q s = t\int_t^\infty p(s)\,\mathrm{d}_q s. \tag{15}$$

The q-L'Hospital rule gives

$$\lim_{t\to\infty} t\int_t^\infty \frac{w^2(s)}{1+(q-1)sw(s)}\,\mathrm{d}_q s = \lim_{t\to\infty}\frac{qt^2w^2(t)}{1+(q-1)tw(t)} = \frac{q[\vartheta_1]_q}{1+(q-1)[\vartheta_1]_q}.$$

Hence, from (15) we get

$$\lim_{t\to\infty} t\int_t^\infty p(s)\,\mathrm{d}_q s = \frac{[\vartheta_1]_q^2 - [\vartheta_1]_q}{1+(q-1)[\vartheta_1]_q} = A.$$

Similar arguments show that also $v \in \mathcal{RV}(\vartheta_2)$ being a positive increasing solution of (9) implies (13).

Note that even without assuming monotonicity, a solution $u \in \mathcal{RV}(\vartheta_1)$ necessarily decreases while a solution $v \in \mathcal{RV}(\vartheta_2)$ necessarily increases by (vi) of proposition 1.

Sufficiency. Let u be a positive decreasing solution of (9) on $[a,\infty)_q$. Then $\lim_{t\to\infty} \mathrm{D}_q u(t) = 0$. Indeed, if not, then there is $K > 0$ such that $\mathrm{D}_q u(t) \leqslant -K$ for $t \in [a,\infty)_q$ since $\mathrm{D}_q u$ is negative increasing. Hence $u(t) \leqslant u(a) - (t-a)K$. Letting $t \to \infty$ we have $\lim_{t\to\infty} u(t) = -\infty$, a contradiction with positivity of u. Integration of (9) from t to ∞ yields $\mathrm{D}_q u(t) = -\int_t^\infty p(s)u(qs)\,\mathrm{d}_q s$. Hence,

$$0 < \frac{-t\mathrm{D}_q u(t)}{u(t)} = \frac{t}{u(t)}\int_t^\infty p(s)u(qs)\,\mathrm{d}_q s \leqslant t\int_t^\infty p(s)\,\mathrm{d}_q s. \tag{16}$$

If (11) holds, then we are done since (16) implies $\lim_{t\to\infty} t\mathrm{D}_q u(t)/u(t) = 0$, and so $u \in \mathcal{SV}_q$. Next we assume (13) with $A > 0$. Set $\eta(t) = t\mathrm{D}_q u(t)/u(t)$. From (16), $0 < -\eta(t) \leqslant t\int_t^\infty p(s)\,\mathrm{d}_q s$, and so η is bounded. Further, η satisfies the modified Riccati q-difference equation

$$\mathrm{D}_q\left(\frac{\eta(t)}{t}\right) - p(t) + \frac{\eta^2(t)/t^2}{1+(q-1)\eta(t)} = 0 \tag{17}$$

with $\eta/t \in \mathcal{R}^+$ on $[a,\infty)_q$. Since η is bounded, we have $\lim_{t\to\infty} \eta(t)/t = 0$ and so integration of (17) from t to ∞ yields

$$-\frac{\eta(t)}{t} = \int_t^\infty p(s)\,\mathrm{d}_q s - \int_t^\infty \frac{\eta^2(s)/s^2}{1+(q-1)\eta(s)}\,\mathrm{d}_q s. \tag{18}$$

Let us write condition (13) as $t\int_t^\infty p(s)\,\mathrm{d}_q s = A + \varepsilon(t)$, where $\lim_{t\to\infty}\varepsilon(t) = 0$. Further, let us write

$$\int_t^\infty \frac{\eta^2(s)/s^2}{1+(q-1)\eta(s)}\,\mathrm{d}_q s = G(t)\int_t^\infty \frac{\mathrm{d}_q s}{qs^2} = \frac{G(t)}{t},$$

where $m(t) \leqslant G(t) \leqslant M(t)$ with $m(t) = \inf_{s\geqslant t} q\eta^2(s)/(1+(q-1)\eta(s))$ and $M(t) = \sup_{s\geqslant t} q\eta^2(s)/(1+(q-1)\eta(s))$. With these equalities, multiplication of (18) by t yields

$$G(t) - \eta(t) = A + \varepsilon(t). \tag{19}$$

We claim that $\lim_{t\to\infty}\eta(t) = [\vartheta_1]_q$. Recall that η is bounded and denote $K_* = \liminf_{t\to\infty}(-\eta(t))$, $K^* = \limsup_{t\to\infty}(-\eta(t))$. Observe the monotone properties of the function $f(x) = qx^2/(1+(q-1)x)$ which occurs in the formula for G. Recall that our 'admissible' x's are just the nonpositive ones satisfying $1+(q-1)x > 0$; the function f

is decreasing for these x's. The function G is bounded. Define K_1 by $\liminf_{t\to\infty} G(t) = qK_1^2\big/(1-(q-1)K_1)$ and K_2 by $\limsup_{t\to\infty} G(t) = qK_2^2\big/(1-(q-1)K_2)$. Thanks to monotonicity of f and boundedness of η we have $0 \leqslant K_* \leqslant K_1 \leqslant K_2 \leqslant K^* < 1/(q-1)$. Now we distinguish several cases which lead to a contradiction, and altogether show that $\lim_{t\to\infty} \eta(t)$ exists and is equal to $[\vartheta_1]_q$. Assume, for instance, $K_1 < -[\vartheta_1]_q$. Then $K_* < -[\vartheta_1]_q$. Noticing that

$$A = \frac{[\vartheta_1]_q^2 - [\vartheta_1]_q}{1+(q-1)[\vartheta_1]_q} = \frac{q[\vartheta_1]_q^2}{1+(q-1)[\vartheta_1]_q} - [\vartheta_1]_q$$

and taking lim inf as $t\to\infty$ in (19) we obtain

$$\frac{qK_1^2}{1+(q-1)(-K_1)} + K_* = \frac{q[\vartheta_1]_q^2}{1+(q-1)[\vartheta_1]_q} - [\vartheta_1]_q.$$

Thanks to monotonicity of f, from the last equation we have $K_* = B - [\vartheta_1]_q$, where $B = f([\vartheta_1]_q) - f(-K_1)$ is positive. Hence, $K_* > -[\vartheta_1]_q$, a contradiction. In a similar manner we obtain a contradiction when $K_* < -[\vartheta_1]_q$ and $K_1 = -[\vartheta_1]_q$. If $K_1 > -[\vartheta_1]_q$, then $K^* \geqslant K_2 > -[\vartheta_1]_q$ and a contradiction is obtained by taking lim sup as $t\to\infty$ in (19). This proves that $\lim_{t\to\infty}\eta(t) = [\vartheta_1]_q$, and so $u(t) = t^{\vartheta_1}L(t) \in \mathcal{RV}_q(\vartheta_1)$, where u is a positive decreasing solution of (9) and $L \in \mathcal{SV}_q$. Now consider a linearly independent solution v of (9), which is given by $v(t) = u(t)\int_a^t (1/(u(s)u(qs)))\,\mathrm{d}_q s$. Put $z = 1/u^2$. Then $z \in \mathcal{RV}_q(-2\vartheta_1)$ by (v) of proposition 1. Since $\int_a^\infty (1/(u(s)u(qs)))\,\mathrm{d}_q s = \infty$, the q-L'Hospital rule and theorem 1 yield

$$\begin{aligned}\lim_{t\to\infty}\frac{t/u(t)}{v(t)} &= \lim_{t\to\infty}\frac{tz(t)}{\int_a^t (1/(u(s)u(qs)))\,\mathrm{d}_q s} = \lim_{t\to\infty}\frac{z(t)+qtD_q z(t)}{1/(u(t)u(qt))}\\ &= \lim_{t\to\infty}\left(\frac{u(t)u(qt)}{u^2(t)} + \frac{qu(t)u(qt)}{u^2(t)}\cdot\frac{tD_q z(t)}{z(t)}\right) = q^{\vartheta_1} + q^{\vartheta_1+1}[-2\vartheta_1]_q =: \gamma.\end{aligned}$$

Hence, $\gamma v(t) \sim t/u(t) = t^{1-\vartheta_1}/L(t)$. Consequently, $v(t) = t^{\vartheta_2}\tilde{L}(t)$, where $\tilde{L}(t) \sim 1/(\gamma L(t))$, $\tilde{L} \in \mathcal{SV}_q$, and so $v \in \mathcal{RV}_q(\vartheta_2)$ by (v) of proposition 1 since $\vartheta_2 = 1 - \vartheta_1$. The last equality follows from

$$\begin{aligned}\vartheta_2 &= \log_q[(q-1)\lambda_2+1] = \log_q[(q-1)(A(q-1)+1-\lambda_1)+1]\\ &= \log_q\left[(q-1)\left(\frac{(q-1)\left(\lambda_1^2-\lambda_1\right)}{1+(q-1)\lambda_1} + 1 - \lambda_1\right) + 1\right]\\ &= \log_q \frac{q}{1+(q-1)\lambda_1}\\ &= \log_q q - \log_q[(q-1)\lambda_1+1]\\ &= 1-\vartheta_1.\end{aligned}$$

The solution v increases by (vi) of proposition 1. For the quantity γ we have

$$\gamma = q^{\vartheta_1}\left(1 + \frac{q^{1-2\vartheta_1}-q}{q-1}\right) = q^{\vartheta_1}\frac{q^{1-2\vartheta_1}-1}{q-1} = q^{\vartheta_1}[1-2\vartheta_1]_q.$$

The theorem is proved. □

Remark 2. (i) Denote the set of all positive solutions of (9) as $\mathbb{M}$. Thanks to the monotonicity, the set $\mathbb{M}$ can be further split in the two classes $\mathbb{M}^+$ and $\mathbb{M}^-$, where

$$\begin{aligned}\mathbb{M}^+ &= \{y \in \mathbb{M} : \exists t_y \in q^{\mathbb{N}_0} \text{ such that } y(t) > 0, \mathrm{D}_q y(t) > 0 \text{ for } t \geqslant t_y\},\\ \mathbb{M}^- &= \{y \in \mathbb{M} : y(t) > 0, \mathrm{D}_q y(t) < 0\}.\end{aligned}$$

Denote $\mathbb{M}^-_{SV} = \mathbb{M}^- \cap \mathcal{SV}_q, \mathbb{M}^-_{RV}(\vartheta_1) = \mathbb{M}^- \cap \mathcal{RV}_q(\vartheta_1), \vartheta_1 < 0, \mathbb{M}^+_{RV}(\vartheta_2) = \mathbb{M}^+ \cap \mathcal{RV}_q(\vartheta_2), \vartheta_2 > 0, \mathbb{M}^-_0 = \{y \in \mathbb{M}^- : \lim_{t\to\infty} y(t) = 0\}, \mathbb{M}^+_\infty = \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = \infty\}$. In view of theorem 2 and proposition 1, we can write

$$\mathbb{M}^- = \mathbb{M}^-_{SV} \iff (11) \iff \mathbb{M}^+ = \mathbb{M}^+_{RV}(1) = \mathbb{M}^+_\infty,$$
$$\mathbb{M}^- = \mathbb{M}^-_{RV}(\vartheta_1) = \mathbb{M}^-_0 \iff (13) \iff \mathbb{M}^+ = \mathbb{M}^+_{RV}(\vartheta_2) = \mathbb{M}^+_\infty.$$

(ii) In the if parts of theorem 2, conditions (11) and (13) can be replaced by the simpler ones $\lim_{t\to\infty} t^2 p(t) \to 0$ and $\lim_{t\to\infty} t^2 p(t) \to A/q$, respectively.

(iii) For related results concerning differential and difference equations cases see [12, 14], respectively. Observe how the constants (indices of regular variation) ϑ_1, ϑ_2 in theorem 2 differ from those in the continuous case. On the other hand, note how theorem 2 resembles the continuous result as $q \to 1$.

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